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Linear Programming: Characteristics, Theory, Methods and Applications.

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Abstract

Linear programming is a mathematical method that is used to determine the best possible outcome or solution from a given set of parameters or list of requirements, which are represented in the form of linear relationships. Characteristics of linear programming include constraints, objective's function, linearity, finiteness, non- negativity, decision variables and data. The theory of linear programming is known as "The General representation Theorem" which will be clearly understood by taking note of the following definitions: "A hyper plane H in $R^{\sim}$ is a set of the form $\{x: px = k\}$ where p is a nonzero vector in $R^{\sim}$ and k is a scalar.", "A half space is a collection of points of the form $\{x: px \geq k\}$ or $\{x: px \leq k\}$", "A polyhedral set is the intersection of a finite number of Half spaces. It can be written in the form $\{x: Ax \leq b\}$ where A is an m x n matrix (where m and n are integers)" and lastly "Let $X \epsilon R^{\sim}$ where n is an integer. The following are the methods of solving linear programming "the linear programming simplex method and the graphical method". The applications of linear programming are and not limited to the following "engineering field, manufacturing industrials, energy industry, transportation optimization etc". The advantages of linear programming are "Linear programming provides insights to the business problem, it helps to solve multi-dimensional problem, it helps to take the best optimal solution. The disadvantages of linear programming are as follows: it takes place only with the variables that are linear, it does not consider change of variables, it cannot solve the nonlinear function and impossible to solve a problem that has more than two variables in the graphical method.

Keywords: Linear programming, methods, applications and constraints.

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1.0 Introduction

1.1 Definitions

Linear Programming (LP) is a powerful technique used for optimization where the objective is to maximize or minimize a linear function subject to a set of linear constraints.Linear Programming is a branch of mathematical optimization that deals with maximizing or minimizing a linear objective function subject to a set of linear constraints. LP has become an essential tool in various fields due to its efficiency and wide range of applications (Vanderbei, R. J., 2008).

This paper provides a comprehensive overview of linear programming, covering its fundamental characteristics, theoretical underpinnings, methodologies, and real-world applications. Additionally, the advantages and limitations of linear programming are discussed to provide a balanced perspective on its utility.

The fundamental components of linear programming include:

i. Decision Variables: These are the unknown quantities that needs to be determined. They represent the things you can control or adjust (e.g., production level, investment amount).
ii. Objective Function: This function illustrates how each decision variable impacts the cost, or simply put, the value that needs to be maximized.
iii. Constraints: These reflect how each decision variable allocates limited resources, and they consist of numerical representations of the relationships between the objective function and the constraints. Constraints typically expressed in linear inequalities or equations.

The role of linear programming in enhancing managerial decisions is crucial. This technique can greatly influence the factors that play a significant role in the decision-making process of any business. It can effectively reduce product costs and optimize the profitability of a business (Bertsimas, D., 1997). Linear programming strategies can be utilized to calculate product prices, ultimately maximizing the company's profits. By employing linear programming techniques, a company can optimize its profits. Additionally, this approach can assist managers in making informed decisions and determining the necessary resources for product production. Such estimations are essential for analyzing the raw material or resource requirements for various products to maximize the company's profits. Furthermore, effective managerial skills are key in maximizing a firm's profitability.

1.2 Assumptions

Before we get too focused on solving linear programs, it is important to review some theory. For instance, several assumptions are implicit in linear programing problems. These assumptions are:

1. **Proportionality** The contribution of any variable to the objective function or constraints is proportional to that variable. This implies no discounts or economies to scale. For example, the value of $8x_1$ is twice the value of $4x_1$, no more or less.
2. **Additivity** The contribution of any variable to the objective function or constraints is independent of the values of the other variables.
3. **Divisibility** Decision variables can be fractions. However, by using a special technique called integer programming, we can bypass this condition. Unfortunately, integer programming is beyond the scope of this paper.
4. **Certainty** This assumption is also called the deterministic assumption. This means

that all parameters (all coefficients in the objective function and the constraints) are known with certainty. Realistically, however, coefficients and parameters are often the result of guesswork and approximation. The effect of changing these numbers can be determined with sensitivity analysis, which will be explored later in Section 9 (Winston 2003).

1.3 Manipulating a Linear Programming Problems

Many linear problems do not initially match the canonical form presented in the introduction, which will be important when we consider the Simplex algorithm. The constraints may be in the form of inequalities, variables may not have a nonnegativity constraint, or the problem may want to maximize z instead of minimize z. We now consider some ways to manipulate problems into the desired form.

1. **Constraint Inequalities** We first consider the problem of making all constraints of a linear programming problem in the form of strict equalities. By introducing new variables to the problem that represent the difference between the left and the right-hand sides of the constraints, we eliminate this concern. Subtracting a slack variable from a "greater than or equal to" constraint or by adding an excess variable to a "less than or equal to" constraint, trans- forms inequalities into equalities. For example, the constraint $4x_1 + x_2 \leq 3$becomes $4x_1 + x_2 + e_1 = 3$ with the addition of $e_1 \geq 0$. If the constraint wereoriginally $4x_1 + x_2 \geq 3$, the additional surplus variable s_1 must be subtracted ($4x_1 + x_2 - s_1 = 3$) so that s_1 can be a strictly nonnegative variable.
2. **URS Variables** If a variable can be negative in the context of a linear programming problem it is called a urs (or "unrestricted in sign") variable.By replacing this variable $x_j$with $x^{J}_{j} - x^{JJ}_{j}$ where $x^{J}_{j}, x^{JJ}_{j} \geq 0$, the problem fits the canonical form.
3. **Minimization/Maximization** If needed, converting a maximization problem to a minimization problem is quite simple. Given that z is an objective function for a maximization problem.

$$\max z = -\min(-z).$$

2.0 Theory and Characteristics of Linear Programming

2.1 Introduction to Optimization Problems:

Optimization is the process of finding the best possible solution among all feasible alternatives to a specific problem. It involves maximizing or minimizing an objective function, subject to a set of constraints, within a given set of variables. The main aim of maximization is to increase efficiency, effectiveness, or performance by making the best use of available resources while adhering to certain limitations and restrictions. It is a fundamental concept min various field, including mathematics, engineering and management. Optimization problems common in real-world scenarios where resources are limited, and decision makers seek to make the most effective and efficient choices. Optimization problems can be classified into two main types:

a. Continuous optimization: in this type of the decision variables can take any real value within a range. The objective function and constraints are continuous mathematical functions.
b. Discrete optimization: this optimization involves problems where the decision variables are restricted to specific discrete values. This type of problem is often encountered in situations where decisions must be made from a finite set of choices.

2.2 Formulation of linear programming problems.

A linear programming problem deals with optimizing (maximizing or minimizing) a function. It constitutes three parts: Objective function, decision variable and constraints. The functions which need to be optimized are known as the objective function. The variables whose value determines the solution are called the decision variables of the problem. The set of simultaneous linear equations or inequalities that the problem is subject to are known as constraints.

In linear programming problems, the term 'linear' refers to the fact that all the variables involved in the objective functions and constraints are linear, that is of degree 1 in the problem under consideration. Then term 'programming' refers to the process of determining a particular cause of action.

The following rule must be applied while formulating a linear programming problem:

a. There must be a well-defined objective to achieve (maximize or minimize).
b. There is only a finite number of decision variables.
c. At least a few of the resources must be on supply, which gives rise to constraints.
d. All the elements should be quantifiable. All the decision variables should assume only non-negative values.
e. Both the given objective function and constraints must be linear equations or inequalities.
f. There must be alternative courses of the line of action to choose from.

2.2.1 Mathematical Formation.

Formation of a linear programming problem refers to translating the real-world problem into the form of mathematical equations which could be solved. It usually requires a thorough understanding of the problem. The steps towards formulating a linear programming problem are highlighted below:

Step 1: Identify the 'n' number of decision variables when given the behavior of the objective function which needs to be optimized.

Step 2: Identify the set of constraints on the decision variables and express them in form of linear equations or inequalities. This will set up our region in the n-dimensional space within which the objective function needs to be optimized. Don't forget to impose the condition of non-negativity on the decision variables. Use all of them must be positive since the problem myths represent a physical scenario, and such variable cannot be negative.

Step 3: Express the objective function on the form of a linear equation in the decision variables.

Step 4: Optimize the objective function either graphically or mathematically.

Example 1; Product Planning

A company products, A and B using three resources: labor hours, raw materials, and machine hours. The company want to maximize the total profit while considering the limited availability of resources.

Solution;

Objective: maximize the total profit.

Decision variables: let X_1 be the number of units of product A to produce, and X_2 be the

number of units product B to produce.

Constraints;

Labor constraints:	$2X_1 + 3X_2 \leq 120$ (hours)
Raw material constraints:	$4X_1 + 2X_2 \leq 100$ (Kg)
Machine hours constraints:	$3X_1 + 2X_2 \leq 90$ (hours)
Non negative constraints:	$X_1 \geq 0; X_2 \geq 0$
Maximize	$z = 5X_1 + 4X_2$

Example 2; Investment portfolio management.
An investor wants to allocate funds among three investment options to maximize the total return (in percentage).
Decision variables:

Let X_1 be the investment option 1,
Let X_2 be the investment option 2,
Let X_3 be the investment option 3.

Constraints:

Budget constraints:	$X_1 + X_2 + X_3 \leq 100{,}000$ ($)
Risk constraints:	$0.2X_1 + 0.3X_2 + 0.1X_3 \leq 20{,}000$
Maximize	$z = 0.15X_1 + 0.12X_2 + 0.18X_3$

2.3 Characteristics of Linear Programming

Linear programming is characterized by several key features:
1. Linear Objective Function: The function to be maximized or minimized is linear.
2. Linear Constraints: The restrictions or limitations imposed on the decision variables are also linear.
3. Non-negativity Restriction: Decision variables are typically required to be non-negative.
4. Feasibility Region: The set of all possible solutions that satisfy the constraints forms a convex polytope.

3.0 Methods to Solve Linear Programming Problems

The linear programming can be solved using different methods, such as the graphical method, simplex method, or by using tools such as R, open solver etc. Here, we will discuss the two most important techniques called the simplex method and graphical method in detail.

A. Linear Programming Simplex Method

The simplex method is one of the most popular methods to solve linear programming problems. It is an iterative process to get the feasible optimal solution. In this method, the value of the basic variable keeps transforming to obtain the maximum value for the objective function. It involves iteratively moving from one corner point to another until an optimal solution is reached (Winston, 2004). The algorithm for linear programming simplex method is provided below:

STEP 1: Establish a given problem. (i.e.,) write the inequality constraints and objective function.

STEP 2: Convert the given inequalities to equations by adding the slack variable to each inequality expression.

STEP 3: Create the initial simplex tableau. Write the objective function at the bottom row. Here, each inequality constraint appears in its own row. Now, we can represent the problem in the form of an augmented matrix, which is called the initial simplex tableau.

STEP 4: Identify the greatest negative entry in the bottom row, which helps to identify the pivot column. The greatest negative entry in the bottom row defines the largest coefficient in the objective function, which will help us to increase the value of the objective function as fastest as possible.

STEP 5: Compute the quotients. To calculate the quotient, we need to divide the entries in the far-right column by the entries in the first column, excluding the bottom row. The smallest quotient identifies the row. The row identified in this step and the element identified in the step will be taken as the pivot element.

STEP 6: Carry out pivoting to make all other entries in column is zero.

STEP 7: If there are no negative entries in the bottom row, end the process. Otherwise, start from step 4.

STEP 8: Finally, determine the solution associated with the final simplex tableau.

B. Graphical Method

The graphical method is used to optimize the two-variable linear programming. If the problem has two decision variables, a graphical method is the best method to find the optimal solution. In this method, the set of inequalities are subjected to constraints. Then the inequalities are plotted in the XY plane. Once, all the inequalities are plotted in the XY graph, the intersecting region will help to decide the feasible region. The feasible region will provide the optimal solution as well as explains what all values our model can take. Let us see an example here and understand the concept of linear programming in a better way.

Example 1:

Calculate the maximal and minimal value of x - 5x + 3y for the following constraints.

$x + 2y \leq 14$

$3x - y \geq 0$

$x - y \leq 2$

Solution: The three inequalities indicate the constraints. The area of the plane that will be marked is the feasible region.

The optimization equation (z) = 5x + 3y You have to find the (x,y) corner points that give the largest and smallest values of z.

To begin with, first solve each inequality.

$x+2y \leq 14 \rightarrow y \leq -(1/2)x + 7$

$3x - y \geq 0 \rightarrow y \leq 3x$

$X - y \leq 2 \rightarrow y \rightarrow x - 2$

Here is the graph for the above equations.

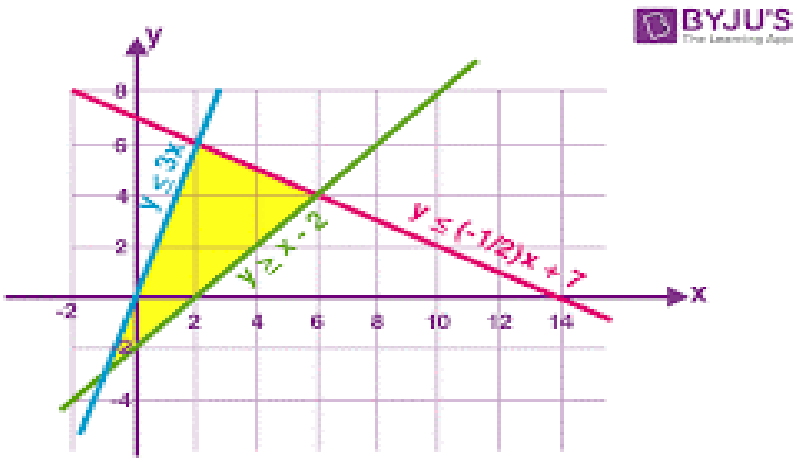


Figure 1: a graph showing the graphical representation of example 1 (byju's learning app. 2022)

Now pair the lines to form a system of linear equations to find the corner points.

y = - (1/2) X + 7

y = 3X

Solving the above equations, we get the corner points as (2,6)

y = - 1/2 X + 7

y = X - 2

Solving the above equations, we get the corner points as (6,4)

y = 3X

y = X - 2

Solving the above equations, we get the corner points as (-1, -3)

For linear systems, the maximum and minimum values of the optimisation equation lie on the comers of the feasibility region. Therefore, to find the optimum solution, you only need to plug these three points in z = 3x + 4y

(2,6):

z = 5(2) + 3(6) = 10 + 18 = 28

(6, 4) :

z = 5(6) + 3(4) = 30 + 12 = 42

(- 1, - 3) :

z = 5(- 1) + 3(- 3) = - 5 - 9 = - 14

Hence, the maximum of lies at (6, 4) and the minimum of z = - 14 lies at (-1, -3)

4.1 Applications, Merits and Demerits of Linear Programming
4.2 Applications.

Linear programming has a vast array of applications across different fields:

i. Operations Research

Production Planning: Optimize the production mix to maximize profit or minimize costs.
Transportation and Logistics: Determine the most efficient routes for shipping goods.

ii. Economics

Resource Allocation: Allocate resources in a way that maximizes utility or profit (Hiller et al., 1010).
Supply and Demand: Balance supply and demand efficiently to optimize pricing strategies.

iii. Engineering

Network Design: Optimize the design and flow within networks (e.g., telecommunications, water distribution).
Energy Management: Optimize the operation of power systems for cost and efficiency.

iv. Finance

Portfolio Optimization: Maximize returns or minimize risks in investment portfolios (Bazaraa et al., 2010).

4.2 Merits and Demerits of Linear Programming

Merits

a. Optimal Solutions: Provides clear optimal solutions within the feasible region.
b. Flexibility: Can be adapted to a wide range of problems across various industries.
c. Efficiency: Modern algorithms and computational tools make solving LP problems fast and efficient.
d. Predictability: Solutions are predictable and reproducible given the linear nature of the models.

Demerits

a. Linearity Assumption: Assumes linearity in both the objective function and constraints, which may not always be realistic.
b. Scalability Issues: Large-scale problems can be computationally intensive.
c. Sensitivity: Solutions can be highly sensitive to changes in the coefficients of the objective function or constraints.
d. Infeasibility and Unboundedness: Problems can be infeasible (no solution) or unbounded (solutions can grow indefinitely).

5.0 Conclusion

Linear programming remains an essential tool in the arsenal of operations research, offering robust solutions to optimization problems across diverse fields. While it has its limitations, the benefits of linear programming often outweigh its drawbacks, particularly when applied to well-defined problems with clear linear relationships.

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